Discrete Exterior Calculus

Lectures by Anil N. Hirani

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Abstract

These notes pertain to a series of lectures on Discrete Exterior Calculus (DEC) given by Prof. Anil Hirani at the University of Oxford, from 31 August to 4 September, while hosted by Prof. Kaibo Hu. The material was taken primarily from my handwritten lecture notes. However, I have tried to elaborate and formalise ideas drawing on the relevant references. In addition, I have added my own perspective whenever it seemed helpful for my own understanding. Lecture 1 sets up the basic objects of DEC, including simplicial complexes, chains and cochains, wedge product, and boundary and coboundary operators. Lecture 2 concerns the averaging interpretation of the wedge product, its identification with Wilson's cochain product through Whitney forms, the failure of associativity, and how A_∞-structures come into the picture.

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1 Lecture 1: simplicial complexes, d, $\wedge$, naturality

As its name suggests, DEC offers a discrete theory mimicking the smooth one. On this note, differential forms become cochains and manifolds become simplicial complexes. The metric-independent part of the theory is then a matter of simplicial algebraic topology[1], and requires no differentiable structure. This implies—as Anil remarked—that little to no knowledge of differential geometry is required to understand DEC. However, it shall be very useful in order to get the discrete–to–smooth analogy.

1.1 Simplicial complexes

Definition 1.1. *Given $k \leq N$, a collection of points $v_0, \ldots, v_k \in \mathbf{R}^N$ is said to be **affinely independent** if the set $\{v_1 - v_0, \ldots, v_k - v_0\}$ is linearly independent in the sense of linear algebra. A (geometric) k-**simplex** is then the set* $\operatorname{conv}\{v_0, \ldots, v_k\}$.

Definition 1.2 (Orientation)**.** *An ordering of the vertices of a simplex determines an **orientation**, denoted $\sigma = [v_0 \cdots v_k]$. Formally, an orientation is a choice of equivalence class in $S_{k+1}/A_{k+1} \cong \mathbf{Z}_2$.*

Since A_{k+1} is the collection of even permutations, two orderings have the same orientation (are equivalent) when they differ by an even permutation; The simplex listed in increasing order carries the canonical positive orientation, and the simplex with reversed orientation is denoted $-\sigma$.

Definition 1.3. *The convex hull f of any subset of the vertices of a simplex σ is again a simplex, and it is called a **face** or **subcell** of σ. This defines a partial ordering $\prec$ in the collection of all simplices in $\mathbf{R}^n$.*

At times, it shall be convenient to consider the abstract approach: a k-simplex σ **is** the set formed by its vertices, and its geometric realisation, or underlying space, is denoted by $|\sigma|$.

Definition 1.4 (Simplicial complex)**.** *A **simplicial complex** is a collection X of simplices such that*

(1) All the faces of simplices in X belong to X;

(2) The intersection $\sigma \cap \sigma'$ of two simplices $\sigma, \sigma' \in X$ is a subcell of σ and σ'.

It can be shown that the second statement may be replaced by: distinct simplices in X have disjoint interiors.

For a simplicial complex X, we denote by X^k the collection of k-simplices. The **k-skeleton** is then defined as

$$X^{(k)} = \bigsqcup_{l=0}^{k} X^l \,.$$

Definition 1.5 (Chains and cochains)**.** *An integer-valued k-**chain** is a formal sum*

$$\sum_{\sigma \in X^k} c_\sigma \, \sigma \,,$$

[1] Hirani's preferred reference on the subject is Munkres' Elements of Algebraic Topology Munkres et al. (2025)

with a finite number of nonvanishing coefficients $c_\sigma \in \mathbf{Z}$. Equivalently, it is a map $X^k \to \mathbf{Z}$ with finite support that, for $\sigma \in X^k$, satisfies $c(-\sigma) = -c(\sigma)$. The collection of these is the additive group $C_k(X;\mathbf{Z})$. In short,

$$C_k(X;\mathbf{Z}) = \mathsf{Free}_{\mathbf{Z}}(X^k)\,.$$

The space of **R***–homomorphisms is denoted*

$$C^k(X;\mathbf{R}) = \mathsf{Hom}_{\mathbf{Z}}\left(C_k(X;\mathbf{Z}),\mathbf{R}\right).$$

The elements of $C^k(X;\mathbf{R})$ are called k-cochains.

We find it relevant to notice that, by construction, X^k generates the group $C_k(X;\mathbf{Z})$. Therefore, to define a k-cochain, it suffices to specify its values on all the k-simplices.

Example 1.1 (de Rham)**.** *Let $|X|$ denote the geometric realisation of the simplicial complex X. For a fixed, locally integrable k-form $\widehat{\alpha} \in \mathrm{L}^1_{\mathrm{loc}}\Omega^k(|X|)$, the map*

$$X^k \ni \sigma \mapsto \int_\sigma \widehat{\alpha}$$

is well-defined. It extends by $\mathbf{Z}$-linearity to a map defined in all of $C_k(X;\mathbf{Z})$, i.e., a k-cochain. The assignment $R : \mathrm{L}^1_{\mathrm{loc}}\Omega^k(|X|) \to C^k(X)$ is called the **de Rham map** *(cf. Section 2.3). The celebrated* ***de Rham theorem*** *states that this map induces an isomorphism in cohomology.*

Definition 1.6 (Boundary)**.** *Fixed a positive integer k, a map $\partial_k : C_k(X) \to C_{k-1}(X)$ is determined by its values on simplices,*

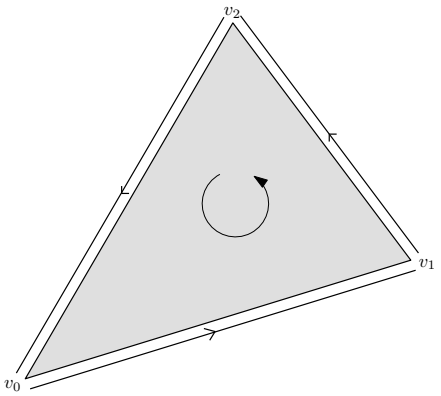
$$\partial_k\,[v_0\cdots v_k] \;=\; \sum_{i=0}^{k}(-1)^i\;[v_0\cdots\widehat{v_i}\cdots v_k]\,,$$

where $\widehat{v_i}$ means that the vertex is omitted. One has $\partial_k\partial_{k+1} = 0$.

Example 1.2. *To justify the previous definition, we compute the boundary of a* 0*-simplex (point),* 1*-simplex (segment) and a* 2*-simplex (triangle), obtaining*

$$\partial_0\,[v_0] = \varnothing\,, \qquad \partial_1\,[v_0v_1] = [v_1] - [v_0]\;, \quad \textit{and} \quad \partial_2\,[v_0v_1v_2] = [v_1v_2] - [v_0v_2] + [v_0v_1]\;.$$

respectively. This coincides with our geometric intuition that the boundary of a point is empty, the boundary of a segment is formed by its endpoints, and the boundary of a triangle is made up of its edges. In addition, the boundary operator encodes the orientation in which we run along the boundary.



1.2 The discrete exterior derivative

From now on, assume the top-dimensional simplices of X are consistently oriented, i.e., whenever two top cells share a face, the orientations induced on it by the cells are opposite. This is equivalent to say that the orientation of top-dimensional cells induces an oriented manifold.

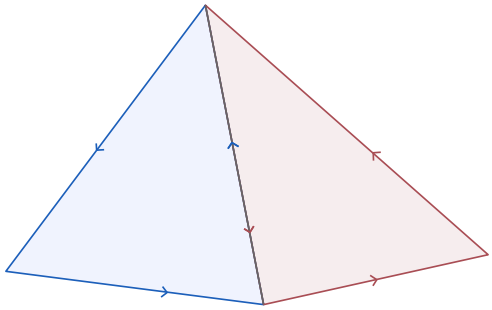


Figure 1: Consistent orientation of a complex formed by two 2-simplices.

Further, suppose the lower-dimensional simplices have been oriented arbitrarily.

Definition 1.7 (Desbrun et al. (2005a), Schubel, Berwick-Evans and Hirani (2024))**.** *The operator* $\mathrm{d}^k : C^k(X) \to C^{k+1}(X)$ *is the adjoint of* ∂_{k+1} *with respect to the evaluation pairing:*

$$\left\langle \mathrm{d}^k \alpha, c \right\rangle := \langle \alpha, \partial_{k+1} c \rangle \,, \qquad \alpha \in C^k(X),\ c \in C_{k+1}(X; \mathbf{Z}).$$

This is the **coboundary operator** of simplicial homology, the identity $\mathrm{dd} = 0$ follows by duality.

Remark 1.1. *The relation in Definition 1.7 is a discrete instance of Stokes theorem. Indeed, if instead of the evaluation pairing, we define a pairing*

$$(\cdot, \cdot) : \Omega^k(|X|) \times C_k(X; \mathbf{Z}) \to \mathbf{R} \quad \textit{by} \quad (\alpha, \sigma) = \int_{|\sigma|} \alpha \,.$$

Then Stokes theorem reads

$$(\mathrm{d}^k \alpha, \sigma) = (\alpha, \partial_k \sigma) \,.$$

Morally, at the discrete level we swap differential forms for cochains and integration for evaluation (of cochains).

Discrete evaluation maps and the corresponding discrete Stokes' theorems are also object of study in the theory of generalised Finite Elements Systems (see, e.g., Christiansen and Hu (2023, Remarks 11,14)).

1.3 The discrete wedge product

Just as we multiply differential forms, we need to multiply cochains. The continuous wedge product $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$ is bilinear, associative and **graded commutative**. The latter meaning

$$\alpha \wedge \beta = (-1)^{\deg \alpha \, \deg \beta} \beta \wedge \alpha \,.$$

The discrete analogue we shall construct retains the first and third properties properties and loses the second one; this is studied in Lecture 2.

In the following discussion, we give some motivation regarding our definition of the discrete wedge product, which may very well differ from Hirani's original motivation:

For a simplicial complex X, a k-cochain α and an l-cochain β, we want to construct a $(k+l)$-cochain $\alpha \curlywedge \beta$. Since cochains are dual objects (homomorphisms), the only piece of data we have at out disposal is their evaluation on chains, so our definition can only involve evaluations of the form $\alpha(\sigma)$ and $\beta(\tau)$, where σ is a k-simplex and τ is an l-simplex[2].

Unfortunately, a $(k+l)$-simplex has $k+l+1$ vertices, and thus it cannot be split into $k+1$ and $l+1$ vertices without repeating one vertex. Thus two choices are to be made: first, we must choose the repeating vertex, and second, we must fix an ordering of the simplices. From the viewpoint of categories, this breaks the naturality of the assignment, i.e., these two choices give raise to a non-canonical assignment. To overcome this, we shall need to add more structure[3]:

Definition 1.8. *A* **locally ordered simplicial complex** *is a simplicial complex X, together with a partial ordering $<$ of the vertex set X^0. We further require that every pair of vertices lying in a common simplex are comparable.*

Using this definition and seeking bilinearity, for any $(k+l)$-simplex $\sigma = [v_0 \cdots v_{k+l}]$, it makes sense to consider a product

$$\langle \alpha, [v_0 \cdots v_k] \rangle \, \langle \beta, [v_k \cdots v_{k+l}] \rangle \ .$$

It is important to notice the similarity between this operation and the tensor product in the smooth setting: they are virtually identical, except for the repeated component. This suggests that the previous operation may be interesting on its own. This is, in fact, the case; the previous operation is the **cup product** defined by *Alexander-Čech-Whitney* (see Munkres et al. (2025, Definition 5.10.1)) for simplicial cohomology. From now on, we denote

$$\smile : C^k(X) \times C^l(X) \to C^{k+l}(X) \,, \quad \langle \alpha \smile \beta, [v_0 \cdots v_{k+l}] \rangle = \langle \alpha, [v_0 \cdots v_k] \rangle \, \langle \beta, [v_k \cdots v_{k+l}] \rangle \ . \tag{1}$$

As a close relative of the tensor product, one can prove that $\smile$ is associative and satisfies a Leibniz rule with d, while graded commutativity fails almost tautologically.

In this situation, to get rid of the arbitrariness of the initial choice of ordering, we need to average over all the possibilities. At the same time, this will repair the lack of graded commutativity. As in the smooth counterpart, the natural way of averaging is an alternation. We have arrived at the following definition (Hirani (2003); Schubel, Berwick-Evans and Hirani (2024)):

$$\langle \alpha \curlywedge \beta, \sigma \rangle = \frac{1}{(k+l+1)!} \sum_{\pi \in S_{k+l+1}} \mathsf{sgn}(\pi) \, \langle \alpha \smile \beta, \pi \cdot \sigma \rangle \ , \tag{2}$$

where $\pi\cdot$ is the natural action of S_{k+l+1} on X^{k+l}. This is referred to as the **DEC wedge product**.

[2] We remind that, to define a cochain, it is enough to specify its value on simplices.

[3] This is indeed more structure. We introduced orientation by fixing an ordering of the vertices, however, this induces an identification of instances such as $[v_0 \cdots v_k] \equiv [v_1 \cdots v_k v_0]$. Being locally ordered is finer than this, and in fact it is the finest relation possible between orderings, i.e., it does not identify any different orderings.

Remark 1.2 (On the normalisation). *The wedge as originally written in Desbrun et al. (2005a, Def. 8.1) carries additional geometric weights in each term. However, the definition used in the lectures has the advantage of having a purely combinatorial form throughout, and it is also the one in Schubel, Berwick-Evans and Hirani (2024); the point is that this is a metric-independent construction.*

To simplify notation, when it does not lead to confusion, we write k-simplices just by specifying $k+1$ tags, for instance $[0 \cdots k]$ is a $(k+1)$-simplex[4]. In addition, it will be useful for computations to denote, for any k-cochain α and any oriented simplex $[i_0 \cdots i_k]$,

$$\alpha_{i_1 \cdots i_k} = \langle \alpha, [i_1 \cdots i_k] \rangle \ .$$

This should not be confused with some type of coefficient as in the case of differential forms.

Example 1.3. *Let X be the complex induced by an edge $[01]$. Then, for all $f \in C^0(X)$ and $\alpha \in C^1(X)$, we have*

$$\langle f \curlywedge \alpha, [01] \rangle = \tfrac{1}{2} \Big[\langle f \smile \alpha, [01] \rangle - \langle f \smile \alpha, [10] \rangle \Big] = \tfrac{1}{2} \big[f_0 \alpha_{01} - f_1 \alpha_{10} \big] = \frac{f_0 + f_1}{2} \, \alpha_{01} \, .$$

The value of the product on the edge is the average *of f over the endpoints, times the value of α. This is an instance of the general averaging interpretation discussed in Lecture 2.*

Example 1.4. *Let X be the complex induced by $[012]$ and $\alpha, \beta \in C^1(X)$. Then (2) reads*

$$\langle \alpha \curlywedge \beta, [012] \rangle = \tfrac{1}{6} \big[\alpha_{01}\beta_{12} - \alpha_{02}\beta_{21} - \alpha_{10}\beta_{02} + \alpha_{12}\beta_{20} + \alpha_{20}\beta_{01} - \alpha_{21}\beta_{10} \big] \, .$$

Collecting by shared vertices,

$$\langle \alpha \curlywedge \beta, [012] \rangle = \tfrac{1}{6} \big[(\alpha_{01}\beta_{02} - \alpha_{02}\beta_{01}) + (\alpha_{01}\beta_{12} - \alpha_{12}\beta_{01}) + (\alpha_{02}\beta_{12} - \alpha_{12}\beta_{02}) \big] \, ,$$

i.e. an alternating product at each of the three vertices. This way of grouping term has been standard since Hirani (2003), and it is an alternative to that in Section 2.2.

Lemma 1.1 (Leibniz rule, Desbrun et al. (2005a); Schubel, Berwick-Evans and Hirani (2024))**.** *For $\alpha \in C^k(X)$ and $w \in C^l(X)$,*

$$\mathsf{d}(\alpha \curlywedge w) \ = \ \mathsf{d}\alpha \curlywedge w + (-1)^k \, \alpha \curlywedge \mathsf{d}w \, .$$

Idea of proof. It suffices to test on a $(k+l+1)$-simplex σ. The action of each $\pi \in S_{k+l+2}$ formally[5] commutes with ∂, so

$$\langle \mathsf{d}(\alpha \curlywedge w), \sigma \rangle = \langle \alpha \curlywedge w, \partial\sigma \rangle = \sum_\pi \mathsf{sgn}(\pi) \, \langle \alpha \smile w, \pi \cdot \partial\sigma \rangle = \sum_\pi \mathsf{sgn}(\pi) \, \langle \mathsf{d}(\alpha \smile w), \pi \cdot \sigma \rangle \ ,$$

where we used the duality beween d and ∂. One then applies the Leibniz rule for the cup product termwise. See Schubel, Berwick-Evans and Hirani (2024, Prop. 3.10) for more details. □

[4] This is formalised as follows: a k-simplex σ is **any** set of $k+1$ elements, together with an embedding $\rho_\sigma : \sigma \to V$, where V is an affine space (e.g. $\mathbf{R}^n$). Its geometric realisation is then $|\sigma| = \mathsf{conv}\,\{\rho_\sigma(0), \ldots, \rho_\sigma(k)\}$. An ordering of σ induces a (manifold) orientation in $|\sigma|$

[5] One must be careful on how π acts on $(k+l)$-chains; this is fixed by regarding it as a simplicial map (see next section).

Example 1.5. *Let $f, g \in C^0(X)$ on the edge $[01]$. The left-hand side is $\langle \mathsf{d}(fg), [01]\rangle = f_1 g_1 - f_0 g_0$, and using Example 1.3,*

$$\langle \mathsf{d}f \curlywedge g, [01]\rangle = \tfrac{1}{2}\left[(f_1 - f_0)g_1 - (f_0 - f_1)g_0\right], \qquad \langle f \curlywedge \mathsf{d}g, [01]\rangle = \tfrac{1}{2}\left[f_0(g_1 - g_0) - f_1(g_0 - g_1)\right],$$

whose sum is $f_1 g_1 - f_0 g_0$.

1.4 What should morphisms be in DEC?

Smooth exterior calculus takes place in the category of manifolds and smooth maps, and naturality of d and $\wedge$ (commutation with pullback by smooth maps) encodes fundamental structure — the chain rule, for instance, is naturality of d. For some years, DEC had specified the objects but not the morphisms of the corresponding category. The proposal of Schubel, Berwick-Evans and Hirani (2024) is that abstract simplicial maps are the right discrete proxy.

Definition 1.9 (Abstract simplicial map)**.** *Given simplicial complexes X, Y, a **vertex map** is a map $\varphi^{(0)} : X^0 \to Y^0$ with the property that it sends (vertices of) simplices of X to simplices of Y. Such a map defines a unique map $\varphi : X \to Y$, called a* **simplicial map***. It can be thought of as a family of maps $\varphi^{(k)} : X^{(k)} \to Y^{(k)}$, because the image of a k-simplex has dimension at most k.*

Remark 1.3. *One must notice that such a map can collapse simplices, i.e., send a simplex to a lower dimensional one. This is the case if and only if the vertex map fails to be injective in some 1-simplex. That is, it collapses an edge. By obvious reasons, the dimension of the image simplex cannot be higher.*

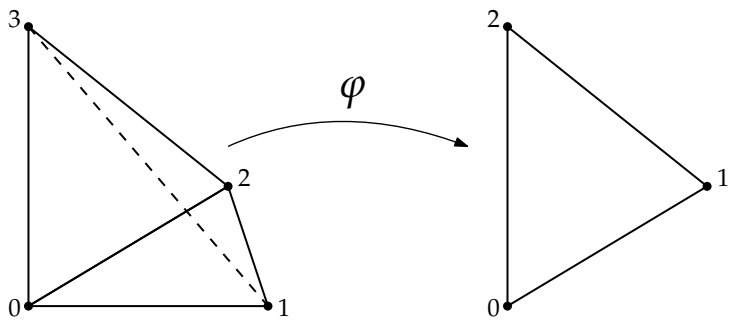


Figure 2: The vertex map sends both 0 and 1 to 0. This collapses $[0\,1]$, and every edge incident in 1.

Remark 1.4. *The collection of simplicial complexes together with simplicial maps as morphisms forms a category.*

In the smooth setting, the assignment of the algebra of differential forms is a contravariant functor, while the assignment of the tangent bundle is a covariant functor. Since our discrete theory considers chains and cochains as replacements for vector fields and differential forms, to define the corresponding functors, first we need that our morphisms induce maps on chains and cochains, with the arrows in the right direction.

Since a simplicial map $\varphi : X \to Y$ sends simplices to simplices, it extends to a map on chains by requiring linearity. However, since a simplicial map can collapse simplices, the resulting map

need not respect the chain degree—just as the original map need not respect the cell dimension. To get rid of these pathological cases, we define

$$\varphi_{\#} : C_k(X;\mathbf{Z}) \to C_k(Y;\mathbf{Z}) \qquad \text{by} \qquad \varphi_{\#}(\sigma) = \begin{cases} \varphi(\sigma)\,, & \text{if } \dim(\sigma) = k\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Remark 1.5. *After this definition, the reader may be tempted (as I was) to restrict attention to simplicial maps that do not collapse simplices. This repairs the natural definition for the induced map on chains, without the need of considering cases.*

The apparent need for a different definition is that, by making a choice in it, we may have broken the functoriality of the assignment. Luckily, this is not the case, and the assignment does provide a covariant functor from the category of simplicial complexes to the category of chain complexes of abelian groups (see 1.1).

In addition, we mention that, from the topological viewpoint, a theory with maps that do not collapse cells is too restrictive, as it does not allow one to map spaces of higher dimension to spaces of lower dimension, even if they carry the same topological information. For example, there is no such map from an edge to a point, although both have trivial (reduced) homology.

The following result implies that the induced maps are chain maps, i.e., morphisms in the category of chain complexes of abelian groups.

Proposition 1.1. *There holds*

$$\partial\varphi_{\#} = \varphi_{\#}\partial\,.$$

Proof. This can be readily checked by using the definitions of $\varphi_{\#}$ and ∂, taking care of the two relevant cases. □

Definition 1.10 (pullback)**.** *The **pullback** $\varphi^* : C^k(Y) \to C^k(X)$ of a simplicial map $\varphi : X \to Y$ is given by*

$$\langle \varphi^*\alpha, c\rangle \;:=\; \langle \alpha, \varphi_{\#}c\rangle\,, \qquad c \in C_k(X)\,.$$

Proposition 1.2 (Naturality, Schubel, Berwick-Evans and Hirani (2024))**.** *Let $\varphi : X \to Y$ be an abstract simplicial map. Then*

$$\varphi^*\mathsf{d}\alpha = \mathsf{d}\,\varphi^*\alpha \quad (\alpha \in C^k(Y)), \qquad \varphi^*(\alpha \curlywedge \beta) = \varphi^*\alpha \curlywedge \varphi^*\beta \qquad \forall\, \alpha \in C^k(Y)\,,\ \beta \in C^l(Y)\,.$$

Comments on the proof. Naturality of d is immediate from the definitions. For the naturality of $\curlywedge$, we refer to Schubel, Berwick-Evans and Hirani (2024). □

Homework 1.1. *Fill out the details of Proposition 1.2.*

2 Lecture 2: wedge, associativity, A_∞ algebras

2.1 A brief motivation for the need of a wedge product

Nonlinear terms in PDEs often involve products of functions, as in $\mathrm{div}(\phi\boldsymbol{u})$ for two-phase flow with ϕ the phase indicator Wang et al. (2023). Less directly, the convective term of the

incompressible Navier–Stokes equations[6], $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$, is the vector proxy of

$$\mathcal{L}_{\boldsymbol{u}}\boldsymbol{u}^\flat - \tfrac{1}{2}\,\mathrm{d}\, i_{\boldsymbol{u}}\boldsymbol{u}^\flat\,,$$

by Cartan's magic formula $\mathcal{L}_{\boldsymbol{u}}\boldsymbol{u}^\flat = \mathrm{d}\, i_{\boldsymbol{u}}\boldsymbol{u}^\flat + i_{\boldsymbol{u}}\,\mathrm{d}\boldsymbol{u}^\flat$, and contraction is itself expressible through the wedge,

$$i_X\alpha \;=\; (-1)^{k(n-k)} \star \big(\star\,\alpha \wedge X^\flat\big), \qquad \alpha \in \Omega^k(M),\ \dim M = n\,.$$

The latter identity is not trivial, and was proved in Desbrun et al. (2005a, Lemma 10.2). This is part of the formulation used in the DEC discretisation of Mohamed, Hirani and Samtaney (2016).

2.2 Averaging interpretation

In Example 1.3 we saw that the product of a 0- and a 1-cochain can be written as a weighted average. The content of Schubel, Berwick-Evans and Hirani (2024) is that this is part of a general principle.

Example 2.1. *Let X be the complex induced by $[012]$ and fix $\alpha, \beta \in C^1(X)$. Collecting terms by edges rather than by vertices, we have*

$$\langle \alpha \curlywedge \beta, [012]\rangle = \frac{1}{3}\left[\frac{\alpha_{20}+\alpha_{21}}{2}\,\beta_{01} + \frac{\alpha_{01}+\alpha_{02}}{2}\,\beta_{12} + \frac{\alpha_{10}+\alpha_{12}}{2}\,\beta_{20}\right].$$

For a geometric interpretation (Figure 3), I borrowed intuition from Berchenko-Kogan and Gawlik (2025), in that we consider the blow-up[7] of the simplex $[012]$. For each edge, we are multiplying the value of β by the average of the two values in the opposite vertex. The three values thus obtained are averaged again to obtain the wedge product.

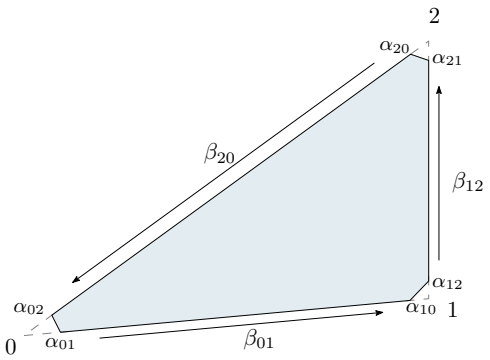


Figure 3: Averaging in a 2-simplex

Homework 2.1. *In the previous example, we multiplied values at each oriented edge, by the averages over the 2 opposite "blown-up-vertices". Use this principle to generalise the previous interpreation of the DEC wedge to simplices of any dimension (see Figure 4 for $k = 3$).*

[6] This example is eerily relevant to current affairs (OpenAI, 2026).

[7] Morally, the blow-up 2-simplex recognises that each vertex has two adjacent edges, and labels them as such. Therefore it has six vertices and edges. Its shape does not matter because the construction is purely combinatorial.

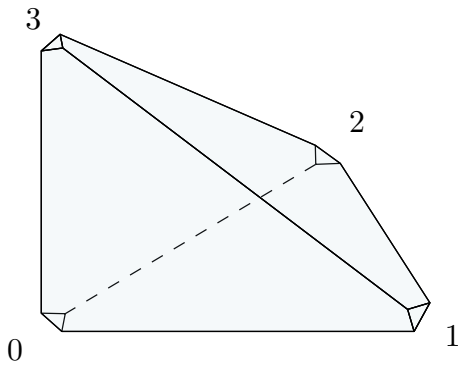


Figure 4: Blow-up 3-simplex

Example 2.2. *For $\alpha \in C^0(X)$ and $\beta \in C^2(X)$ on the triangle, $\langle \alpha \curlywedge \beta, [012] \rangle = \left(\frac{\alpha_0 + \alpha_1 + \alpha_2}{3}\right)\beta_{012}$.*

For a face f of a simplex σ, we denote by $\sigma \backslash f$ the simplex obtained by deleting the vertices in f from σ, with the induced orientation. If g is an oriented simplex and v a vertex, $v * g$ is an oriented simplex formed by joining v with the vertices of g.

Proposition 2.1 (Averaging interpretation)**.** *Consider $\alpha \in C^k(X)$, $\beta \in C^l(X)$ and $\sigma \in X^{k+l}$. Then*

$$\begin{aligned}\langle \alpha \curlywedge \beta, \sigma \rangle &= \frac{1}{\binom{k+l+1}{k+1}} \sum_{f^k \prec \sigma} \langle \alpha, f \rangle \left(\frac{1}{k+1} \sum_{v^0 \prec f} \langle \beta, v * (\sigma \setminus f) \rangle \right) \\ &= \frac{1}{\binom{k+l+1}{l+1}} \sum_{f^l \prec \sigma} \left(\frac{1}{l+1} \sum_{v^0 \prec f} \langle \alpha, v * (\sigma \setminus f) \rangle \right) \langle \beta, f \rangle \ ,\end{aligned}$$

where $f^k \prec \sigma$ ranges over the k-faces of σ, and the orientations are chosen so that the ordering $(f \setminus \{v\}, v, \sigma \backslash f)$ represents an even permutation in S_{k+l+1}.

Proof. See Schubel, Berwick-Evans and Hirani (2024) for a disentanglement of the content of the result through several examples, and in particular Proposition 4.7 for a proof. □

With d read as a difference operator and $\curlywedge$ as a double average, a consequence Anil drew is that the metric-independent part of DEC consists of purely arithmetic operations, with makes DEC a very accessible framework.

2.3 FEEC interpretation

Definition 2.1 (Whitney map)**.** *Let X be a simplicial complex triangulating an oriented manifold. We can define a map $\mathrm{W} : C^k(X) \to \mathcal{P}_1^- \Lambda^k(X)$ as follows: on the basis cochain σ^* dual to $\sigma = [v_0 \cdots v_k]$, i.e., $\langle \sigma^*, \tau \rangle = \delta_{\sigma\tau}$,*

$$\mathrm{W}\sigma^* \ = \ k! \sum_{i=0}^{k} (-1)^i \, \lambda_{v_i} \, \mathrm{d}\lambda_{v_0} \wedge \cdots \wedge \widehat{\mathrm{d}\lambda_{v_i}} \wedge \cdots \wedge \mathrm{d}\lambda_{v_k} \, ,$$

λ_v being the barycentric coordinate attached to v.

We owe the reader unfamiliar with FEEC and simplicial algebraic topology an explanation of the previous definition. The motivation comes from *Whitney forms*, first defined by Whitney Whitney (1957). They solve the problem of, starting from a cochain, constructing the simplest affine differential form possible. You may consult Appendix A for a brief summary of this construction, starting from low-dimensional cells to understand the principle.

Proposition 2.2. *The Whitney map is the unique linear right inverse of the de Rham map mapping $C^k(X)$ into $\mathcal{P}_1\Lambda^k(X)$. Moreover, its range satisfies*

$$\mathrm{W}(C^k(X)) = \mathcal{P}_1^-\Lambda^k(X)\,,$$

the space of trimmed polynomial differential forms of lowest degree Arnold, Falk and Winther (2010).

Remark 2.1. *By this reason, via Whitney forms, DEC is closely related with FEEC. This has recently been made explicit, with two independent groups establishing convergence of DEC for the Hodge–Laplacian in 2025 (Zhu, Christiansen, Hu and Hirani, 2025; Guzmán and Potu, 2025).*

Proposition 2.3. *Let $\alpha \in C^k(X)$, $\beta \in C^l(X)$. Then,*

$$\forall\,\tau \in X^{k+l} \;:\; \langle \alpha \curlywedge \beta, \tau\rangle \;=\; \int_\tau \mathrm{W}\alpha \wedge \mathrm{W}\beta\,,$$

equivalently

$$\alpha \curlywedge \beta = \mathrm{R}\big(\mathrm{W}\alpha \wedge \mathrm{W}\beta\big)\,.$$

Proof. See Schubel, Berwick-Evans and Hirani (2024, Prop. 2.5). □

Remark 2.2 (Wilson's cochain product)**.** *The right-hand side in the previous identity is **Wilson's cochain product** Wilson (2007, Definition 5.1). The proof hinges on Theorem 5.2 of the same article. This identification had not been made explicit in previous DEC literature, and it has two main implications. First, by Wilson (2007, Thm. 5.4) the combinatorial DEC wedge converges to the smooth wedge under refinement. Second, the proposition connects DEC to the literature on algebraic structures on simplicial cochains: by a theorem of Sullivan (1977) (see also Tradler, Zeinalian and Sullivan (2007) and Wilson (2007, Thm. 5.11)) there is a local construction extending the algebra of DEC canonically to a C_∞-algebra, which under refinement converges to the algebra of forms Wilson (2007, Thm. 5.12). That suggests a concrete route to a more accurate combinatorial wedge: include more of the higher operations.*

2.4 Properties of the DEC wedge, and the failure of associativity

So far, it has been established that the the DEC wedge is graded commutative, is functorial (Proposition 1.2), and satisfies a Leibniz rule (Lemma 1.1).

Associativity of the DEC wedge fails, as the following example shows:

Example 2.3 (Failure of associativity)**.** *Let $X = [01]$, $f, g \in C^0(X)$ and $\alpha \in C^1(X)$. Using the averaging form,*

$$\langle (f \curlywedge g) \curlywedge \alpha, [01]\rangle = \frac{f_0 g_0 + f_1 g_1}{2}\,\alpha_{01}\,, \qquad \langle f \curlywedge (g \curlywedge \alpha), [01]\rangle = \frac{f_0 + f_1}{2}\cdot\frac{g_0 + g_1}{2}\,\alpha_{01}\,,$$

and these are not equal: the average of a product is not the product of averages. With $f_0 = 1, f_1 = 0$, $g_0 = 0, g_1 = 1$ *and* $\alpha_{01} = 1$ *one gets* 0 *and* $\frac{1}{4}$.

Instead, one can only prove associativity on cocycles (cochains whose discrete exterior derivative is zero) Desbrun et al. (2005a, Lemma 8.3).

Example 2.4 (Associativity restored on cocycles)**.** *Take the same* $X = [01]$, $f, g \in C^0(X)$, $\alpha \in C^1(X)$ *as in Example 2.3. Since* X *has no* 2*-simplices,* $C^2(X) = 0$*, so every* 1*-cochain — in particular* α *— is automatically closed. Now also take* f, g *closed, i.e. constant:* $f_0 = f_1 = f$ *and* $g_0 = g_1 = g$*. Then*

$$\langle (f \curlywedge g) \curlywedge \alpha, [01] \rangle = \frac{fg + fg}{2} \alpha_{01} = fg\, \alpha_{01}\,, \qquad \langle f \curlywedge (g \curlywedge \alpha), [01] \rangle = \frac{2f}{2} \cdot \frac{2g}{2} \alpha_{01} = fg\, \alpha_{01}\,,$$

and the two bracketings agree, illustrating the general statement above with all three cochains closed.

This has an interesting consequence, appearing in Desbrun et al. (2005a, Rem. 8.1).

Remark 2.3. *The discretisation of a constant smooth form is a closed cochain. Hence on a mesh of characteristic length* Δx*,*

$$\frac{1}{|\sigma^{k+l+m}|} \left\langle \alpha \curlywedge (\beta \curlywedge \gamma) - (\alpha \curlywedge \beta) \curlywedge \gamma, \sigma^{k+l+m} \right\rangle = \mathcal{O}(\Delta x)\,:$$

the associativity defect averages to something of the order of the mesh size and vanishes in the limit.

Remark 2.4 (wedge product in FEEC)**.** *By Proposition 2.3, once one projects a product of Whitney forms back to cochains, the resulting product is no longer associative. One of Anil's remarks was that this may be avoided by testing against suitable functions instead of projecting. There is also the structural difficulty that degrees add under products of polynomials, so the finite element space of* $\alpha \wedge \beta$ *differs from that of the factors—unless the spaces consist of piecewise constant polynomials, in which case the cochain representations are closed. General wedge products have not yet appeared in the FEEC literature outside the special case used to define inner products. This was also remarked in the introduction of (Schubel, Berwick-Evans and Hirani, 2024).*

2.5 Differential graded algebras; CCP and ACP

Definition 2.2 (DGA)**.** *A* ***differential graded algebra*** *is a graded vector space* $V = \bigoplus_{k \in \mathbf{Z}} V^k$ *equipped with a differential* d *and an associative, graded commutative product* $\wedge$ *satisfying a Leibniz rule. This is*

$$\text{(i)}\ \ \mathrm{d} \circ \mathrm{d} = 0, \quad \text{(ii)}\ \ \mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge \mathrm{d}\beta \qquad \text{(iii)}\ \ (\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)\,.$$

Regarding the product as an operator is what brings A_∞ into the picture. Let[8]

$$F \;=\; V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots \;=\; \bigoplus_{k \geq 1} T^k(V)\,,$$

[8] Anil did not include the base field in the sum below, but if one does, then F is the full tensor coalgebra and the "standard" Fock space.

with the reduced **tensor coalgebra**[9] structure. One extends d and $\wedge$ to all of F as coderivations. In particular, this means that, on $V \otimes V$, $\widehat{\mathrm{d}}$ satisfies,

$$\widehat{\mathrm{d}}(\alpha \otimes \beta) = -\big[(\mathrm{d}\alpha \otimes \beta) + (-1)^{|\alpha|}(\alpha \otimes \mathrm{d}\beta)\big]\,,$$

and on $V \otimes V \otimes V$, $\widehat{\Lambda}$ satisfies

$$\widehat{\Lambda}(\alpha \otimes \beta \otimes \gamma) = -\Lambda(\alpha, \beta) \otimes \gamma + \alpha \otimes \Lambda(\beta, \gamma) = -(\alpha \wedge \beta) \otimes \gamma + \alpha \otimes (\beta \wedge \gamma)\,.$$

Writing d, Λ for the extended operators, the three DGA axioms are checked to become[10]

$$\text{(i) } \mathrm{d}^2 = 0, \qquad \text{(ii) } \mathrm{d}\Lambda + \Lambda\mathrm{d} = 0, \qquad \text{(iii) } \Lambda^2 = 0\,. \tag{3}$$

Indeed, for (ii), we have

$$\begin{aligned}(\mathrm{d}\Lambda + \Lambda\mathrm{d})(\alpha \otimes \beta) &= \mathrm{d}(\alpha \wedge \beta) - \Lambda\big[\mathrm{d}\alpha \otimes \beta + (-1)^{|\alpha|}\alpha \otimes \mathrm{d}\beta\big] \\ &= \mathrm{d}(\alpha \wedge \beta) - \big[\mathrm{d}\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge \mathrm{d}\beta\big]\,,\end{aligned}$$

which is zero exactly when the Leibniz rule holds; and $\Lambda^2 = 0$ on $T^3(V)$ is precisely associativity. So a DGA is exactly the statement $(\mathrm{d} + \Lambda)^2 = 0$ together with graded commutativity.

In this situation, the three instances of algebras we have encountered then compare as follows:

	$\mathrm{d}^2 = 0$	$\mathrm{d}\Lambda + \Lambda\mathrm{d} = 0$	$\Lambda^2 = 0$	graded comm.
$(\Omega^*(M), \mathrm{d}, \wedge)$	✓	✓	✓	✓
$(C^\bullet(X), \mathrm{d}, \smile)$	✓	✓	✓	× (CCP)
$(C^\bullet(X), \mathrm{d}, \curlywedge)$	✓	✓	×	✓ (ACP)

The smooth de Rham complex is a DGA, the cochain complex with the cup product is an associative DGA in every respect except graded commutativity, and the cochain complex with the DEC wedge is graded commutative but not associative. The challenge of repairing the second algebra, i.e., obtaining graded commutativity without losing the good properties, is called the **commutative cochain problem** (CCP). This is briefly discussed in Kotiuga (2008), together with a second obstruction he calls the *discrete star localisation problem*, both as concrete instances in which, according to him, the DEC programme cannot be completed. My understanding is that Anil called "ACP" the problem of fixing the third row of the table, this should stand for the **associative cochain problem**.

2.6 A_∞ structures

The notion is due to Stasheff (1963) and Keller (2001) is a readable introduction available online.

[9] The precise meaning of this is somewhat technical and unnecessary for the point we try to make here. Roughly, we endow F with the structure of a (non-counital) coalgebra by considering deconcatenation $\Delta : F \to F \otimes F$ as comultiplication. Anil called F the **Fock space** and pointed out that these constructions were originally developed to study loop spaces

[10] We relax notation and denote $\widehat{\mathrm{d}}$ and $\widehat{\Lambda}$ as d and Λ, respectively.

2.6.1 Motivation: loop spaces and associahedra

Let Y be a topological space and $y_0 \in Y$. The **based loop space** is

$$\Omega(Y, y_0) = \{\gamma : [0,1] \to Y \text{ continuous, } \gamma(0) = \gamma(1) = y_0\},$$

with concatenation

$$(\alpha\beta)(t) = \begin{cases} \alpha(2t), & 0 \le t \le \frac{1}{2}, \\ \beta(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

Concatenation of *parametrised* loops is not associative: $(\alpha\beta)\gamma$ and $\alpha(\beta\gamma)$ traverse the same path with different reparametrisations. The reparametrisation sending one to the other is a homotopy.

In this way, for three factors, the two possible bracketings $(ab)c$ and $a(bc)$, and the path joining them, form an interval with endpoints; the **associahedron** K_3 (see Figure 2.6.1). The situation is similar for four factors; there are five possible bracketings and the resulting is the pentagon K_4 with vertices

$$((ab)c)d, \quad (a(bc))d, \quad a((bc)d), \quad a(b(cd)), \quad (ab)(cd),$$

whose edges are essentially the homotopies coming from K_3.

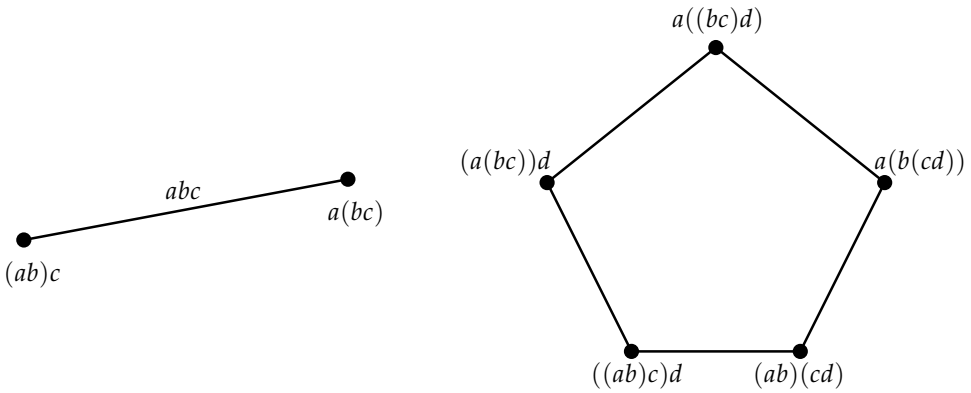


Figure 5: Associahedrons K_3 and K_4

In general K_n records the ways of bracketing n factors, and an **A_∞-structure** is a coherent system of such higher homotopies.

In this sense, we can say that concatenation in the based loop space is homotopy-associative, which is to say associative up to homotopy.

2.6.2 Nilpotency formulation

Anil commented that the physicists' approach in the BRST–BV formalism[11] is to deform $\mathrm{d} + \Lambda$ to a series

$$D = \underbrace{\mathrm{d}}_{\text{unary}} + \underbrace{\Lambda}_{\text{binary}} + \underbrace{m_3}_{\text{ternary}} + m_4 + \cdots$$

[11] This stands for Becchi-Rouet-Stora-Tyutin and Batalin-Vilkovisky formalism.

and require $D^2 = 0$. Sorting by arity one obtains higher order relations

$$\begin{aligned} \mathrm{d}^2 &= 0, \\ \Lambda\mathrm{d} + \mathrm{d}\Lambda &= 0, \\ m_3\mathrm{d} + \Lambda^2 + \mathrm{d}m_3 &= 0, \\ m_4\mathrm{d} + m_3\Lambda + \Lambda m_3 + \mathrm{d}m_4 &= 0, \\ &\vdots \end{aligned} \tag{4}$$

Each equation involves only operators already constructed, together with a single unknown. This is the construction of Dolotin, Morozov and Shakirov (2008), which was sent to Anil by its authors and realises such a structure explicitly on arbitrary simplicial complexes.

The third equation, written as

$$\mathrm{d}m_3 + m_3\mathrm{d} = -\Lambda^2 ,$$

is a homotopy identity: it means that Λ^2, which measures the failure of associativity, is null-homotopic, so the product is associative *up to the homotopy* m_3. Setting $m_1 = \mathrm{d}$, $m_2 = \Lambda$, the general step reads

$$\mathrm{d}m_k + m_k\mathrm{d} \;=\; -M_k , \tag{5}$$

where M_k is built from $m_2, \dots, m_{k-1}$ only. Several routes to solving (5) exist: Stasheff's formula, the DMS lift formula, and the so-called suspension trick which Anil attributed to Keller (2001).

2.6.3 The Poincaré operator

The route taken in the lecture solves (5) with a contracting homotopy. The idea is given by:

Lemma 2.1. *Suppose P is an operator of degree -1 with $\mathrm{d}P + P\mathrm{d} = \mathsf{id}$, and let y commute with d. Then $x = Py$ solves $\mathrm{d}x + x\mathrm{d} = y$.*

Proof. $\mathrm{d}(Py) + (Py)\mathrm{d} = \mathrm{d}Py + P\mathrm{d}y = (\mathrm{d}P + P\mathrm{d})y = y$, where the middle step used commutation. The same computation gives $x = yP$. □

We call such P a **Poincaré operator**[12]. Anil traced the name, in this generality, to Christiansen's work. The two relevant papers that touch this are Christiansen and Hu (2018); Christiansen, Hu and Sande (2020). Though, one must certainly mention that the purpose for which Poincaré operators are used in these papers is completely different.

One of Bingyan Liu's contributions in his PhD thesis, was to construct Poincaré operators in the context of DEC (Guzmán, Hirani, Liu and Potu (2026)). See also Remark 2.5.

Note that the hypothesis of Lemma 2.1 is met at the first nontrivial step: from $\mathrm{d}\Lambda + \Lambda\mathrm{d} = 0$ one gets $\mathrm{d}\Lambda^2 = -\Lambda\mathrm{d}\Lambda = \Lambda^2\mathrm{d}$, so Λ^2 commutes with d and $m_3 = -P\Lambda^2$ solves the third equation of (4). The procedure keeps going recursively.

[12] In homological algebra, such a map is called a cochain homotopy; a homotopy in the category of cochain complexes.

Remark 2.5. *Discrete Poincaré operators on cochains and Whitney forms have recently been constructed under different hypotheses on the complex or the domain as well as a discrete* ***Bogovskiĭ operator*** *preserving homogeneous boundary conditions, in Guzmán, Hirani, Liu and Potu (2026). This paper also records that "discrete" Poincaré operators are key in the recursive construction of an A_∞-structure on cochains over contractible domains, following Dolotin, Morozov and Shakirov (2008). The details are in Liu's thesis Liu (2026).*

3 Lecture 3: vector bundle-valued DEC

3.1 Bundle-valued forms in smooth geometry

As in the scalar-valued case, we first recall the structures that we want to discretise. The smooth theory here gets a bit heavier; for this I find the exposition in Tu (2017) particularly friendly. We apologise in advance for some sloppiness on the notation; we work with several types of forms (e.g. form-valued, bundle-valued, scalar-valued), and not always specify which is the one we mean. This should not lead to confusion, as it should be clear from the context. We believe, at the risk of being terse, that this will help a concise presentation.

Motivation. Anil motivated the coming definitions by considering form-valued forms or double forms, i.e. , smooth sections of $\Lambda^k(T^*M) \otimes \Lambda^l(T^*M)$—as these lead naturally to the problem of a bundle-valued DEC. Such objects may be regarded as $(0, k+l)$-tensor fields alternating both in their first k and last l indices. We shall denote

$$\Omega^{k,l}(M) = \Gamma(\Lambda^k(T^*M) \otimes \Lambda^l(T^*M)) \,.$$

We notice that, via the maps $\alpha \mapsto \alpha \otimes 1$ and $\beta \mapsto 1 \otimes \beta$, we have identifications

$$\Omega^l(M) \cong \Omega^{0,l}(M) \quad \text{and} \quad \Omega^k(M) \cong \Omega^{k,0}(M) \,.$$

The choice of a connection ∇ on TM induces a exterior derivative on $\Omega^{k,l}(M)$, which gives a sequence

$$\Omega^l(M) \;\simeq\; \Omega^{0,l}(M) \xrightarrow{\nabla} \Omega^{1,l}(M) \xrightarrow{d_\nabla} \Omega^{2,l}(M) \xrightarrow{d_\nabla} \cdots \longrightarrow \Omega^{n,l}(M) \,.$$

In general, when E is a vector bundle, we can define E-valued k-forms as follows:

Definition 3.1 (bundle-valued forms)**.** *For $E \to M$ a smooth vector bundle, an E-**valued** k-**form** is an element of*

$$\Omega^k(M; E) \;:=\; \Gamma\big(\Lambda^k T^*M \otimes E\big) \,.$$

So, taking $E := \Lambda^l T^*M$ recovers the previous setting as a special case.
The bilinear map[13]

$$\wedge : \Omega^k(M; E) \times \Omega^l(M) \to \Omega^{k+l}(M; E), \qquad (\alpha \otimes s) \wedge \beta := (\alpha \wedge \beta) \otimes s \,, \tag{6}$$

[13]The reader is warned about the repeated abuse of the symbol $\wedge$ to denote generalisations of the wedge product.

gives $\Omega^\bullet(M;E)$ a $\Omega^\bullet(M)$-module structure[14].

A **connection**[15] $\nabla^E : \Gamma(E) \to \Omega^1(M;E)$ on E determines a degree-one map

$$d^k_{\nabla^E} : \Omega^k(M;E) \to \Omega^{k+1}(M;E)$$

called the **exterior covariant derivative**, by requiring $d_{\nabla^E}|_{\Omega^0(M;E)} \equiv \nabla^E$ and a Leibniz rule:

$$d_{\nabla^E}(\alpha \wedge w) \;=\; d_{\nabla^E}\alpha \wedge w + (-1)^k \alpha \wedge \mathrm{d}w\,, \tag{7}$$

for $\alpha \in \Omega^k(M;E)$ and $w \in \Omega^l(M)$. When E is the trivial line bundle with the trivial connection, d_{∇^E} coincides with the exterior derivative d and we obtain the ordinary Leibniz rule.

This construction gives a sequence

$$\Gamma(E) \simeq \Omega^0(M;E) \xrightarrow{\nabla^E} \Omega^1(M;E) \xrightarrow{d_{\nabla^E}} \Omega^2(M;E) \xrightarrow{d_{\nabla^E}} \cdots \longrightarrow \Omega^n(M;E)\,. \tag{8}$$

In general (8) need not be a complex, and indeed $d_{\nabla^E} \circ d_{\nabla^E}$ is the **curvature** of the connection. For instance, when E is the (trivial) line bundle endowed with a connection $\nabla = \mathrm{d} + \alpha\wedge$, induced by $\alpha \in \Omega^1(M)$, we get $d_\nabla \circ d_\nabla = d\alpha\wedge$, which is clearly nonzero when α is not closed.

On the other hand, going back to the general picture, by applying (7) twice together with $\mathrm{d} \circ \mathrm{d} = 0$, one obtains

$$(d_{\nabla^E} \circ d_{\nabla^E})(\alpha \wedge \beta) = (d_{\nabla^E} \circ d_{\nabla^E})(\alpha) \wedge \beta\,,$$

whence, putting $F^k_{\nabla^E} := d^{k+1}_{\nabla^E} \circ d^k_{\nabla^E}$, we obtain $F_{\nabla^E}(\alpha \wedge \beta) = F_{\nabla^E}(\alpha) \wedge \beta$.

In particular, this implies that $F^k_{\nabla^E} \in \mathsf{Hom}_{C^\infty(M)}(\Omega^k(M;E), \Omega^{k+2}(M;E))$. For $k = 0$, we have

$$\begin{aligned} F^0_{\nabla^E} &\in \mathsf{Hom}_{C^\infty(M)}(\Gamma(E), \Omega^2(M;E)) \cong \Gamma(\mathsf{Hom}(E, \Lambda^2 T^*M \otimes E)) \cong \Gamma(E^* \otimes \Lambda^2 T^*M \otimes E) \\ &\cong \Gamma(\Lambda^2 T^*M \otimes E^* \otimes E) \cong \Gamma(\Lambda^2 T^*M \otimes \mathsf{End}(E)) = \Omega^2(M;\mathsf{End}(E))\,. \end{aligned}$$

Under this identification, one then verifies that for arbitrary k we have

$$F^k_{\nabla^E}(\alpha) = F^0_{\nabla^E} \wedge \alpha\,.$$

Here, we have let $\Omega^\bullet(M;\mathsf{End}(E))$ act on $\Omega^\bullet(M;E)$, in the natural way:

$$\wedge : \Omega^k(M;\mathsf{End}(E)) \times \Omega^l(M;E) \to \Omega^{k+l}(M;E)\,, \qquad (\alpha \otimes A) \wedge (\beta \otimes s) = (\alpha \wedge \beta) \otimes A(s)\,.$$

In addition, we notice that $\Omega^\bullet(M;\mathsf{End}(E))$ acts on itself via

$$\wedge : \Omega^k(M;\mathsf{End}(E)) \times \Omega^l(M;\mathsf{End}(E)) \to \Omega^{k+l}(M;\mathsf{End}(E))\,, \qquad (\alpha \otimes A) \wedge (\beta \otimes B) = (\alpha \wedge \beta) \otimes (AB)\,,$$

thus obtaining the structure of a unital graded associative algebra.

[14] This choice of multiplication seems arbitrary. However, since fiberwise we need a map $\cdot : (A \otimes V) \times A \to A \otimes V$, where $(A, \wedge)$ is a graded algebra and V a vector space, we realise that this is nothing but the construction of the free A-module on V, which is canonical and given by $(a \otimes v) \cdot a' := (a \wedge a') \otimes v$.

[15] We recall that a connection $\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ may be regarded as an $\Omega^1(M;E)$-valued map, as for $s \in \Gamma(E)$, we have $\nabla s \in \mathsf{Hom}_{C^\infty(M)}(\Gamma(TM), \Gamma(E)) \cong \Gamma(\mathsf{Hom}(TM, E)) \cong \Gamma(T^*M \otimes E) = \Omega^1(M;E)$.

So far, besides $\Omega^\bullet(M)$ we have encountered two relevant structures derived from M, E and the choice of connection ∇^E:

$$\Omega^\bullet(M;E) \quad \text{and} \quad \Omega^\bullet(M;\mathsf{End}(E))\,,$$

each carrying its own exterior covariant derivative. One should notice that this construction does not require a metric on M.

The relevant point here is that this construction is **natural**: a smooth map between manifolds $f : N \to M$ determines a pullback bundle f^*E over N, and $f^* : \Omega^\bullet(M;E) \to \Omega^\bullet(N;f^*E)$ commutes with the exterior covariant derivatives for the pullback connection $f^*\nabla^E$. In particular the curvature of ∇^E pulls back to the curvature of $f^*\nabla^E$.

Remark 3.1 (Local structure and gauge transformations)**.** *Relative to a local trivialisation $E|_U \simeq U \times \mathbf{R}^r$ over an open $U \subset M$, the covariant derivative and curvature admit local expressions*

$$d_{\nabla^E}\big|_U \overset{\text{locally}}{=} \mathrm{d} + A, \qquad F\big|_U \overset{\text{locally}}{=} F_A := \mathrm{d}A + A \wedge A, \qquad A \in \Omega^1\big(U;\mathsf{End}(E|_U)\big)\,,$$

*in terms of the **connection** 1**-form** A. A change of local trivialisation $g : U \to \mathsf{GL}_r(\mathbf{R})$ is a **gauge transformation**, under which*

$$A \mapsto gAg^{-1} - \mathrm{d}g\,g^{-1}, \qquad F_A \mapsto gF_Ag^{-1}\,.$$

Remark 3.2. *As mentioned earlier, when $E = \Lambda^l(T^*M)$, we obtain double forms. Relevant examples of double forms are different instances of curvature on manifolds, as well as metrics seen as $(1,1)$-forms. The discretisation of such structures has been focus of much attention during the last years, as this is an important step towards the completion of the* **Finite Element Tensor Calculus** *(FETC) programme. A relevant work in this matter is Berchenko-Kogan and Gawlik (2025), which addresses the discretisation of such objects, mostly in the lowest order case. More recently, Hu and Lin (2025) have provided a canonical construction of finite element spaces of form-valued forms, for arbitrary spatial dimension, form degree and polynomial order, using the now well-known BGG construction at the discrete level. The goal in the remaining of these notes is to identify and construct a discrete counterpart of the bundle-valued setting within DEC, as a first step towards a theory of discrete bundles not derived from sheaves. The latter remark refers to the fact that the existent theory of discrete vector bundles constructed in Christiansen and Hu (2023)—which Anil calls CH-bundles—relies on the local-to-global passage characterising finite element systems (FES), which are precisely presheaves in the category of cochain complexes. From the author's viewpoint, this is not the case of DEC, as the emphasis is put mainly into the discrete structures rather than their globalisations and functional properties. This difference is only moral, as Berwick-Evans, Hirani and Schubel (2021) proves that CH-bundles can be obtained through a coarsening procedure from the theory here described. Furthermore, the functional analysis of DEC (e.g. convergence and stability) has so far needed of an auxiliary theory such as FEEC or FES to be carried out (see Zhu, Christiansen, Hu and Hirani (2025); Guzmán and Potu (2025)).*

3.2 Discrete theory

We expect the discrete theory to enjoy all the properties discussed in the previous section (see Berwick-Evans, Hirani and Schubel (2021, Thm. 1.5) for a precise statement). The strategy shall be to discretise the base manifold and define a connection in the form of parallel transport. Then build cochains, exterior covariant derivative, curvature and other objects up from there. There will be a crucial obstruction along the way, namely curvature, together with the question of when sections integrate globally.

Here are some highlights stated by Anil on the board:

- Products become interesting. The pairings $\Omega^\bullet(M)$ acting on $\Omega^\bullet(M;E)$, $\Omega^\bullet(M;\mathsf{End}(E))$ acting on $\Omega^\bullet(M;E)$, and $\Omega^\bullet(M;\mathsf{End}(E))$ acting on itself, need to get their own discrete construction.
- Stokes' theorem does not always hold.
- Convexity/alternation points to other frameworks: an alternative discretisation, due to Braune et al. (2024), composes the discrete exterior covariant derivative with an averaging (alternation) operator to obtain convergence results, at the cost of obscuring some of the algebraic structure discussed below.
- Antisymmetrisation has a curvature obstruction: on a tetrahedron, requiring the Leibniz rule to hold for every permutation appearing in the antisymmetrised sum forces the curvature on each constituent triangle to vanish. The only viable product between scalar-valued and bundle-valued cochains is therefore the (non-antisymmetrised) cup product Berwick-Evans, Hirani and Schubel (2021, Rem. 3.19, §5.2).
- When $F_\nabla = 0$: the resulting complex computes E-twisted cohomology, with **densities** (sections of the orientation line bundle) as a distinguished example, giving a discrete twisted Poincaré duality Berwick-Evans, Hirani and Schubel (2021, §6, Ex. 6.7).

3.3 Subdivision and local order

We work with **abstract** simplicial complexes throughout this subsection, so that subdivision is a purely set-theoretic construction, with no need to fix in advance how its new vertices are geometrically realised.

The discrete constructions to come are stated for a locally ordered simplicial complex (Lecture 1); the corresponding morphisms are the **order-preserving** simplicial maps.

Definition 3.2 (Order-preserving map)**.** *A simplicial map $\varphi : X \to Y$ between locally ordered simplicial complexes is **order preserving** if $u < v$ implies $\varphi(u) \leq \varphi(v)$, for any two vertices u, v of X lying in a common simplex.*

It is worth recording why an arbitrary simplicial map need not respect any given local order.

Example 3.1. *For $u_0 < u_1 < u_2$ the vertices of a triangle and $v_0 < v_1$ the vertices of an edge, and consider the simplicial map φ with $u_0 \mapsto v_0$, $u_1 \mapsto v_1$, $u_2 \mapsto v_0$. Independently of the local order on the target, φ cannot be made order preserving, as seen in Figure 6.*

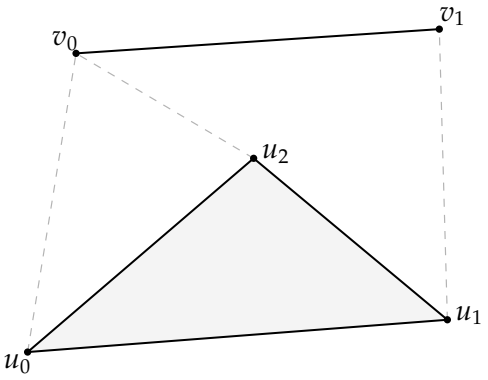


Figure 6: A simplicial map which is not order-preserving.

Definition 3.3 (Subdivision)**.** *For a simplicial complex $\widetilde{X}$, its **subdivision** $X := \mathrm{sd}\,\widetilde{X}$ is the simplicial complex whose vertices are the simplices of $\widetilde{X}$. The set $(X, \prec)$ is a poset, and the k-simplices of the subdivision $\mathrm{sd}\,\widetilde{X}$ are chains of $(k+1)$ elements following this order. Namely,*

$$\left\{\sigma^0 \prec \sigma^1 \prec \cdots \prec \sigma^n\right\}, \qquad 0 \leq i \leq k\,.$$

Remark 3.3 (Realising the abstract subdivision)**.** *Nothing in the previous definition commits one to a particular geometric realisation of $\mathrm{sd}\,\widetilde{X}$: it makes sense to realise a vertex $\sigma \in \widetilde{X}$ as any point associated to σ, typically an interior one, such as a barycentre when one is defined. On this note, there is no canonical geometric realisation for $\mathrm{sd}\,\widetilde{X}$.*

Example 3.2 (Subdivision of a triangle)**.** *The complex $\widetilde{X}$ induced by a triangle $[0\,1\,2]$ has cells*

$$\widetilde{X} = \left\{\, \{0\}\,,\, \{1\}\,,\, \{2\}\,,\, \{0,1\}\,,\, \{1,2\}\,,\, \{0,2\}\,,\, \{0,1,2\} \,\right\}.$$

By definition, this is the vertex set of the subdivision $\mathrm{sd}\,\widetilde{X}$. The inclusion allows one to regard $\tilde{X}$ as a poset, so that the k-simplices are $(k+1)$-uples of elements following this partial order. Dropping the curly brackets, we have

$k=0$	$k=1$	$k=2$
0	$0 \prec 01$	$0 \prec 01 \prec 012$
1	$0 \prec 02$	$0 \prec 02 \prec 012$
2	$1 \prec 01$	$1 \prec 01 \prec 012$
01	$1 \prec 12$	$1 \prec 12 \prec 012$
12	$2 \prec 02$	$2 \prec 12 \prec 012$
02	$2 \prec 12$	$2 \prec 02 \prec 012$.
012	$0 \prec 012$	
	$1 \prec 012$	
	$2 \prec 012$	
	$01 \prec 012$	
	$12 \prec 012$	
	$02 \prec 012$	

Figure 7 shows a reasonable choice of realisation for this abstract simplicial complex, according to the previous remark.

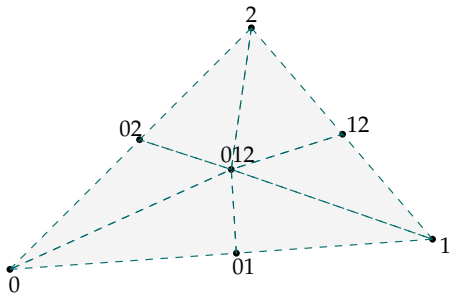


Figure 7: A realisation of sd $\widetilde{X}$

Proposition 3.1 (Berwick-Evans, Hirani and Schubel (2021))**.** sd $\widetilde{X}$ *is canonically a locally ordered simplicial complex, ordering the vertices of each new simplex by the dimension of the corresponding old simplex. Moreover, for any simplicial map* $\varphi : \widetilde{X} \to \widetilde{Y}$*, the induced map* sd φ : sd $\widetilde{X} \to$ sd $\widetilde{Y}$ *is order-preserving.*

Essentially, this means that sd is a covariant functor, sending unordered complexes to locally ordered complexes, and simplicial maps to order-preserving simplicial maps.

So although a local order cannot always be found compatibly with a given simplicial map, as the non-example above shows, one can always be supplied, functorially, after passing to a subdivision. This device allows us to build the discrete theory of vector bundles with connection on arbitrary (unordered) simplicial complexes.

A Whitney forms

For a 0-cochain $\alpha \in C^0(X)$, we need to build a function f_α, defined in $|X|$, by using only its evaluation on vertices. Each vertex $v \in X^0$ has one associated piecewise-affine function $\lambda_v : \mathbf{R}^n \to \mathbf{R}$ satisfying $\lambda_v(v') = \delta_{vv'}$, called the **barycentric coordinate** associated with v. So, if v^* is the 0-cochain dual to v, it makes sense to define

$$\mathrm{W}v^* = \lambda_v \,.$$

Notice that, under this choice

$$\left\langle \mathrm{R}(\mathrm{W}v^*), v' \right\rangle = \left\langle \mathrm{R}(\lambda_v), v' \right\rangle = \lambda_v(v') = \delta_{vv'} = \left\langle v^*, v' \right\rangle \,,$$

as integration on 0-chains reduces to evaluation. This proves that $\mathrm{R}^0\mathrm{W}^0$ is the identity on 0-chains. Namely, the Whitney map is a right inverse for the de Rham map.

For $k = 1$, we need to assign to each edge a 1-form, only by using evaluation on 1-chains. An edge $\sigma = [v_0 v_1]$ has two associated barycentric functions $\lambda_{v_0}, \lambda_{v_1}$, which can be combined to yield a globally defined 1-form

$$r_0 \lambda_{v_1} \mathrm{d}\lambda_{v_0} + r_1 \lambda_{v_0} \mathrm{d}\lambda_{v_1} \,,$$

where r_0, r_1 are real coefficients we must fix. As we shall see later, the property we encountered in form degree zero has many good consequences, so it is worth aiming at preserving it. We force this by choosing r_0 ,r_1 cleverly. Namely, we require

$$\langle \mathrm{R}(r_0 \lambda_{v_1} \mathrm{d}\lambda_{v_0} + r_1 \lambda_{v_0} \mathrm{d}\lambda_{v_1}), \tau \rangle = \delta_{\sigma\tau} \,,$$

which is equivalent to

$$r_0 \int_\tau \lambda_{v_1} \mathrm{d}\lambda_{v_0} + r_1 \int_\tau \lambda_{v_0} \mathrm{d}\lambda_{v_1} = \delta_{\sigma\tau} \,.$$

First, notice

$$\mathrm{d}(\lambda_{v_0}\lambda_{v_1}) = \lambda_{v_0} \mathrm{d}\lambda_{v_1} + \lambda_{v_1} \mathrm{d}\lambda_{v_0} \,,$$

whence

$$\int_\tau (\lambda_{v_0} \mathrm{d}\lambda_{v_1} + \lambda_{v_1} \mathrm{d}\lambda_{v_0}) = \int_\tau \mathrm{d}(\lambda_{v_0}\lambda_{v_1}) = \lambda_{v_0}(v_1)\lambda_{v_1}(v_1) - \lambda_{v_0}(v_0)\lambda_{v_1}(v_0) = 0 \,.$$

Therefore

$$\int_\tau \lambda_{v_1} \mathrm{d}\lambda_{v_0} = -\int_\tau \lambda_{v_0} \mathrm{d}\lambda_{v_1} \,.$$

Thus, recalling $\sigma = \{v_0 + t(v_1 - v_0) \mid t \in [0,1]\}$—parametrised from v_0 to v_1— we obtain

$$\int_\sigma \lambda_{v_0} \mathrm{d}\lambda_{v_1} = \int_0^1 (1-t)\mathrm{d}t = 1 - \frac{1}{2} = \frac{1}{2} \,.$$

Thus, a necessary condition is

$$r_1 - r_0 = 2 \,.$$

In turn, since 1-cochains are compatible with orientation, we have $(-\sigma)^* = -\sigma^*$, and therefore need

$$\mathrm{W}((-\sigma^*)) = -\mathrm{W}\sigma^* = -r_0\lambda_{v_1}\mathrm{d}\lambda_{v_0} - r_1\lambda_{v_0}\mathrm{d}\lambda_{v_1}\,.$$

On the other hand, since we have swapped the vertices, and want our choice of 1-form to hold for every cochain, we also should have

$$\mathrm{W}((-\sigma^*)) = r_0\lambda_{v_0}\mathrm{d}\lambda_{v_1} + r_1\lambda_{v_1}\mathrm{d}\lambda_{v_0}\,.$$

Both conditions yield

$$r_0 + r_1 = 0\,.$$

Therefore $r_0 = -1$ and $r_1 = 1$. We have proved that our choice is consistent for all 1-chains, in the sense that it integrates to 1, independently of orientation. Now, to prove that RW is the identity, we notice that if σ and τ share only one vertex, say v_1, the later must be of the form $[v_1 v_2]$. Since $\lambda_{v_0} \equiv 0$ at both v_1 and v_2, λ_{v_0} vanishes along τ. Hence, we have proved that in this case

$$-\int_\tau \lambda_{v_1}\mathrm{d}\lambda_{v_0} = \int_\tau \lambda_{v_0}\mathrm{d}\lambda_{v_1} = 0\,.$$

The same argument applies when σ and τ do not have any vertices in common. Hence

$$\langle \mathrm{R}(r_0\lambda_{v_1}\mathrm{d}\lambda_{v_0} + r_1\lambda_{v_0}\mathrm{d}\lambda_{v_1}), \tau\rangle = 0\,,$$

for all $\tau \in X^1$, $\tau \neq \sigma$. In particular, we have proved that if we define

$$\mathrm{W}^1\sigma^* = \lambda_{v_0}\mathrm{d}\lambda_{v_1} - \lambda_{v_1}\mathrm{d}\lambda_{v_0}\,,$$

then $\mathrm{R}^1\mathrm{W}^1$ is the identity on 1-chains. Moreover, this is the only choice of r_0 and r_1 that works.

The same construction is extended to higher order by using the wedge product to raise the form degree.

The space of Whitney k-forms coincides with the lowest order trimmed space $\mathcal{P}_1^-\Lambda^k(|X|)$ of FEEC.

AI use declaration. The entirety of these notes was written by the author starting from his handwritten lecture notes. However, AI was used to find some relevant literature not previously known to the author.

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