DUAL SPACE INDUCED BY AN ISOMORPHISM.

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Abstract. This short note concerns with the construction of an instance of the so–called *spaces of negative norm* or *extrapolated spaces*. We adapt a construction due to *Y. M. Berezanskiĭ* in [Ber68, Ch. 3] so as to be able to extend isomorphisms between Hilbert spaces, without losing invertibility. The main difference with respect to Berezanski's approach is that we do not use the (adjoint of the) inclusion map to *pivot* the spaces, but rather the appropriate restriction of the relevant isomorphism. The author is not aware of any existing references for this specific construction, and so all the details are provided.

Consider Hilbert spaces $(H_0, \langle\ ,\ \rangle_0)$ and $(H_2, \langle\ ,\ \rangle_2)$ with continuous inclusion $H_2 \hookrightarrow H_0$, and let $A_0 : H_2 \to H_0$ denote an isomorphism of Hilbert spaces. Denote the induced continuous bilinear form by $a : H_2 \times H_0 \to \mathbf{R}$. The objective is then to, given an intermediate space H_1, define a space $H_{1,a}$, which allows us to "extend" a to an invertible continuous bilinear form in $H_{1,a} \times H_1$, and that inherits compactness properties. In addition, it is desirable that this space identifies in a *nice* way with the *true* dual space of H_1, in a sense to be made precise later. Lastly, we let $\mathfrak{i}_{k,l}$ denote the inclusion map $H_k \hookrightarrow H_l$, for $2 \geq k \geq l \geq 0$; which we require to be continuous and, moreover, we require that $\mathfrak{i}_{1,0}$ is dense (but still strict).

The initial setting can be summarized in the following diagram:

$$H_2 \xhookrightarrow{\ \mathfrak{i}_{2,1}\ } H_1 \xhookrightarrow{\ \mathfrak{i}_{1,0}\ } H_0, \qquad A_0 : H_2 \to H_0 \tag{1.1}$$

Since $\mathfrak{i}_{1,0}$ and a are continuous, given $u \in H_2$, the map $v \mapsto a(u, v)$ is continuous in the H_1-norm. This, according to the *Riesz representation theorem*, gives a linear and bounded map $A_1 : H_2 \to H_1$ characterized by

$$a(u, v) = \langle A_1 u, v \rangle_1 \qquad \forall\, (u, v) \in H_2 \times H_1.$$

We shall define the sought space as the completion of H_2 in a suitable topology, defined in terms of A_1. We then extend A_1 by continuity to get the desired "extension" of A_0.

Let us define a new scalar product in H_2 by requiring that A_1 is an *isometry*: for $u, v \in H_2$, we put

$$\langle u, v \rangle_{1,a} := \langle A_1 u, A_1 v \rangle_1 . \tag{1.2}$$

To see that it is a scalar product, we just need to check homogeneity: given $u \in H_2$ such that $\langle u, u\rangle_{1,a} = 0$, we have that $A_1 u = 0$, and therefore

$$\langle A_0 u, v\rangle_0 = a(u, v) = 0 \qquad \forall\, v \in H_1.$$

Since H_1 is dense in H_0 and A_0 is invertible, we get $u = 0$. We remark that this also proves injectivity of A_1.

We denote by $H_{1,a}$ the completion of H_2 under the induced norm $\|\cdot\|_{1,a}$. First, we check that this completion is nontrivial—in that H_2 is not complete with respect to $\|\cdot\|_{1,a}$— and thereby H_2 is strictly embedded in $H_{1,a}$. The following constitutes the main part of the proof for this fact:

Lemma 1.1. *$(H_2, \|\cdot\|_{1,a})$ is complete if and only A_1 has closed range.*

Proof. Assume that H_2 is complete with respect to $\|\cdot\|_{1,a}$, and consider a sequence $(A_1 u_k)_k$ convergent to z in H_1. In particular, this implies that it is Cauchy, thereby

$$\|u_k - u_\ell\|_{1,a} = \|A_1 u_k - A_1 u_\ell\|_1 \xrightarrow{k,\ell} 0.$$

Therefore, by assumption, there is $u \in H_2$ such that $\|u_k - u\|_{1,a} \xrightarrow{k} 0$, which, according to the definition of $\|\cdot\|_{1,a}$ and the fact that a normed space is Hausdorff, implies that $A_1 u = z$, hence A_1 has closed range.

Conversely, if A_1 has closed range, we have $\|u\|_2 \preceq \|A_1 u\|_1 = \|u\|_{1,a}$, for all $u \in H_2$. Then, a Cauchy sequence $(u_k)_k$ in H_2 with respect to $\|\cdot\|_{1,a}$ is also Cauchy with respect to $\|\cdot\|_2$. Since H_2 is complete, there exists $u \in H_2$ such that $\|u_k - u\|_2 \xrightarrow{k} 0$. From the continuity of A_1 and the definition of $\|\cdot\|_{1,a}$, we get that $(u_k)_k$ converges to u in $\|\cdot\|_{1,a}$. Hence H_2 is complete with this norm. □

Now, we shall see that A_1 cannot have closed range. To this end, notice that

$$\langle u, A_0^\star v\rangle_2 = \langle A_0 u, v\rangle_0 = a(u, v) = \langle A_1 u, v\rangle_1 = \langle u, A_1^\star v\rangle_2\,,$$

for all $(u, v) \in H_2 \times H_1$. There follows that

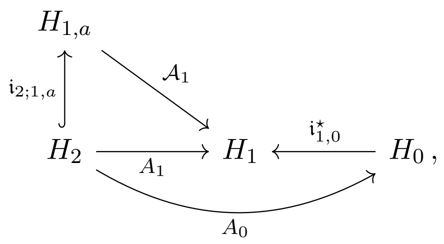
$$A_1^\star = A_0^\star \circ \mathfrak{i}_{1,0}. \tag{1.3}$$

Since A_1 is injective, it has closed range if and only if $A_1^\star$ is surjective. From the fact that A_0 is bijective and (1.3), we get that $A_1{}^\star$ is surjective if and only if $H_1 = H_0$, which is not the case, as we assumed a strict inclusion. Thus, H_2 is a proper subspace of $H_{1,a}$. This fact ensures that the foregoing discussion is not trivial.

In this situation, as previously announced, A_1 extends to a continuous linear map $\mathcal{A}_1 : H_{1,a} \to H_1$, in the standard way, and we may pass to the limit in (1.2) to get

$$\langle u, v\rangle_{1,a} = \langle \mathcal{A}_1 u, \mathcal{A}_1 v\rangle_1 \qquad \forall\, u, v \in H_{1,a}. \tag{1.4}$$

It is clear from the definitions of A_1 and $\mathcal{A}_1$ that the latter extends the former, and so, bearing in mind (1.3), we have arrived at the following commutative diagram:



(1.5)

which relates A_0, A_1 and $\mathcal{A}_1$.

Moreover, if we define $a' : H_{1,a} \times H_1 \to \mathbf{R}$ by putting, for $u \in H_{1,a}$,

$$a'(u, v) = \langle \mathcal{A}_1 u, v \rangle_1 \,,$$

we notice that a and a' are bound to coincide in $H_2 \times H_1$; **this is the "extension" property we were looking for**. Moreover, by construction, we have

$$|a'(u, v)| \leq \|\mathcal{A}_1 u\|_1 \, \|v\|_1 = \|u\|_{1,a} \, \|v\|_1 \,, \tag{1.6}$$

for $(u, v) \in H_{1,a} \times H_1$.

We claim that a' is invertible, which is equivalent to say that $\mathcal{A}_1$ is an isomorphism. In fact, from (1.4), we see that $\mathcal{A}_1$ is injective and has closed range. To see that it is surjective, we assume otherwise; then, we have a nontrivial orthogonal decomposition

$$H_1 = \mathsf{Im}\,\mathcal{A}_1 \oplus \mathsf{Im}\,\mathcal{A}_1^\perp \,.$$

An element v in the second space, in particular, satisfies

$$0 = \langle v, \mathcal{A}_1 u \rangle_1 = a'(u, v) = a(u, v) = \langle v, A_0 u \rangle_0 \,,$$

for all $u \in H_2$. Since A_0 is bijective, this gives $v = 0$, which contradicts the nontrivial decomposition. We conclude that $\mathcal{A}_1$ is an isomorphism.

We remark that (1.6), together with the fact that $\mathcal{A}_1$ is invertible, implies that a' may be regarded as a non-degenerate duality pairing between $H_{1,a}$ and H_1. Furthermore, from (1.6), we also see that every $u \in H_{1,a}$ induces a functional $v \mapsto a'(u, v)$ which lies in the *true* dual space $H_1^\star$. Conversely, by *Riesz representation theorem*, a functional $f \in H_1^\star$ admits a unique $u_f \in H_1$, such that $f(v) = \langle u_f, v \rangle_1$. Since $\mathcal{A}_1$ invertible there is a unique $z_f \in H_{1,a}$, such that $\mathcal{A}_1 z_f = u_f$, and so,

$$a'(z_f, v) = f(v),$$

for all $v \in H_1$. Moreover, we have that $\|z_f\|_{1,a} = \|u_f\|_1 = \|f\|$. Thus, we have described an isometric identification (isomorphism) between $H_{1,a}$ and $H_1^\star$.

Lastly, we shall prove that, if $\mathfrak{i}_{1,0}$ is compact, then $\mathfrak{i}_{2;1,a} : H_2 \hookrightarrow H_{1,a}$ is compact as well. Dualizing (1.3), we deduce that

$$A_1 = ({A_0}^\star \circ \mathfrak{i}_{1,0})^\star = \mathfrak{i}^\star_{1,0} \circ A_0.$$

We notice that $\mathfrak{i}^\star_{1,0}$, being the adjoint of a compact operator, is compact as well. So, the previous identity shows compactness of $A_1 : H_2 \to H_1$.

We shall see that this is equivalent to say that the inclusion $\mathfrak{i}_{2;1,a} : H_2 \to H_{1,a}$ is compact. In fact, according to *Banach-Alaoglu theorem*, a bounded sequence $(u_n)_n$ in H_2 has a (not reindexed) weakly convergent subsequence. Then, A_1 upgrades this to strong convergence of $(A_1 u_n)_n$ in H_1, it follows that

$$\|u_k - u_\ell\|_{1,a} = \|A_1 u_k - A_1 u_\ell\|_1 \ ,$$

which, in virtue of the completeness of $H_{1,a}$, gives convergence of $(u_n)_n$ in $H_{1,a}$. This is the definition of a compact operator.

We remark that one also could have started by directly defining $A_1 = \mathfrak{i}^\star_{1,0} \circ A_0$, in which case everything follows in an analogous way. We chose to present it in this way because it feels less terse and more intuitive.

We summarize this discussion in the following result, for later reference:

Proposition 1.1. *Consider a sequence of continuous inclusions $H_2 \hookrightarrow H_1 \hookrightarrow H_0$ of Hilbert spaces, with the second one being dense. Let $A_0 : H_2 \to H_0$ be an isomorphism with induced bilinear form $a : H_2 \times H_0 \to \mathbf{R}$. Then, there exists a Hilbert space $H_{1,a}$ and an isomorphism $\mathcal{A}_1 : H_{1,a} \to H_1$, with induced bilinear form $a' : H_{1,a} \times H_1 \to \mathbf{R}$, such that*

i) *for all $(u, v) \in H_2 \times H_1$, there holds $a'(u, v) = a(u, v)$,*
ii) *$H_{1,a}$ and $H_1^\star$ are isometrically isomorphic, and*
iii) *if the inclusion $H_1 \hookrightarrow H_0$ is compact, then $H_2 \hookrightarrow H_{1,a}$ is compact as well.*

Remark 1.1. When studying the eigenmodes of a given operator, it is of utmost interest to have a compact inclusion so that one is able to apply the following theorem (see [Bre11, Theorem 6.8]): *"the spectrum of a compact operator is either zero, finite, or an infinite set accumulating at zero"*. Moreover, if our operator is symmetric, then its eigenvectors form a Hilbert basis of the underlying (separable) Hilbert space.

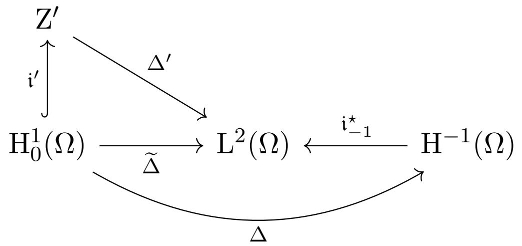
To illustrate a situation where Proposition 1.1 is useful, we outline the following toy example: let us consider an open and connected, bounded domain Ω of $\mathbf{R}^n$. We remind that the distributional Laplacian $\Delta : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ is an isomorphism. Moreover, by virtue of the classical Rellic-Kondrachov theorem, $\mathrm{H}^1_0(\Omega)$ is compactly embedded into $\mathrm{L}^2(\Omega)$, and thus, by duality, we have that the inclusion $\mathrm{L}^2(\Omega) \hookrightarrow \mathrm{H}^{-1}(\Omega)$ is compact.

In this situation, the following diagram is the analogue of (1.1)

$$\begin{array}{ccccc} H^1_0(\Omega) & \overset{\mathfrak{i}_{2,1}}{\hookrightarrow} & \mathrm{L}^2(\Omega) & \overset{\mathfrak{i}_{1,0}}{\hookrightarrow} & H^{-1}(\Omega) \\ & \searrow & \underset{\Delta}{\longrightarrow} & \nearrow & \end{array} \tag{1.7}$$

Now, according to Proposition 1.1, taking $\mathrm{L}^2(\Omega)$ as our intermediate space, there exists a space Z' and an isomorphism $\Delta' : \mathrm{Z}' \to \mathrm{L}^2$ such that $\mathfrak{i}' : \mathrm{H}^1_0(\Omega) \hookrightarrow \mathrm{Z}'$ is compact .

The situation is summarized by the following commutative diagram:



. (1.8)

Furthermore, there holds

$$(\Delta' u, v)_{0,\Omega} = (\Delta u, v)_{-1;\Omega} \qquad \forall\, u \in \mathrm{H}^1_0(\Omega), \quad v \in \mathrm{L}^2(\Omega)\,.$$

Now the application. In finite element methods, one is given a sequence $(\mathrm{H}_n)_n$ of closed subspaces of $\mathrm{H}^1_0(\Omega)$, which are pointwise approximating, this is then used to define[1] the "discrete" operator $\Delta_n : \mathrm{H}_n \to \mathrm{H}^{-1}(\Omega)$ via

$$\langle \Delta_n u, v\rangle_{-1,\Omega} = \langle \Delta u, v\rangle_{-1,\Omega} \qquad \forall\, u, v \in \mathrm{H}_n\,.$$

In this situation, one wishes to prove that the eigenmodes of Δ_n converge to those of Δ, as $n \to \infty$.

To this end, we assume that Δ_n is uniformly invertible: there is $\beta > 0$ such that

$$\inf_{u\in \mathrm{H}_n} \sup_{v\in \mathrm{H}_n} \frac{\langle \Delta_n u, v\rangle_{-1,\Omega}}{\|v\|_{\mathrm{H}^1_0(\Omega)}\|u\|_{\mathrm{H}^{-1}(\Omega)}} \geq \beta\,. \tag{1.9}$$

Therefore, $\Delta_n^{-1} : \mathrm{H}^{-1}(\Omega) \to \mathrm{H}^1_0(\Omega)$ is well-defined.

Since $\Delta^{-1} : \mathrm{H}^{-1}(\Omega) \to \mathrm{H}^{-1}(\Omega)$ is a compact, self-adjoint operator, by virtue of the classical Babuška-Osborn, the desired eigenmode convergence is implied by convergence in operator norm. More specifically, we need that

$$\|\Delta_n^{-1} - \Delta\|_{\mathrm{H}^{-1}(\Omega)\to\mathrm{H}^{-1}(\Omega)} \xrightarrow{n\to\infty} 0\,.$$

This holds, and the proof only hinges on Proposition 1.1 and (1.9).

Proof. Define the energy projector $Q_n : \mathrm{Z}' \to \mathrm{H}_n$ by requiring that, for $u \in \mathrm{Z}'$, there holds

$$\big\langle \Delta' u, v\big\rangle_{-1,\Omega} = \langle \Delta_n Q_n u, v\rangle_{-1,\Omega} \qquad \forall\, v \in \mathrm{H}_n\,. \tag{1.10}$$

We remark that these projectors are well-defined because Δ' and Δ_n are isomorphisms.

Firstly, we can use the inf-sup condition (1.9) to prove that $(Q_n)_n$ is uniformly bounded $\mathrm{Z}' \to \mathrm{H}^{-1}(\Omega)$:

$$\|Q_n u\|_{-1,\Omega} \leq \beta^{-1} \sup_{v\in\mathrm{H}_n} \frac{\langle \Delta_n Q_n u, v\rangle_{-1,\Omega}}{\|v\|_{\mathrm{H}^1_0(\Omega)}} = \sup_{v\in\mathrm{H}_n} \frac{\langle \Delta' u, v\rangle_{-1,\Omega}}{\|v\|_{\mathrm{H}^1_0(\Omega)}} \preceq \|u\|_{\mathrm{Z}'}$$

[1]This is a somewhat unfortunate notation; Δ_n does not have anything to do with the *n-Laplacian*.

Likewise, given $u \in \mathrm{H}_0^1(\Omega)$ and $(u_n)_n \in (\mathrm{H}_n)_n$ converging to u in H^{-1}, we write

$$\|Q_n u - u_n\|_{-1,\Omega} \preceq \sup_{v\in \mathrm{H}_n} \frac{\langle \Delta_n Q_n u - \Delta_n u_n, v\rangle_{-1,\Omega}}{\|v\|_{\mathrm{H}_0^1(\Omega)}} = \sup_{v\in \mathrm{H}_n} \frac{\langle \Delta u - \Delta u_n, v\rangle_{-1,\Omega}}{\|v\|_{\mathrm{H}_0^1(\Omega)}} \preceq \|u - u_n\|_{-1,\Omega}\,,$$

which proves that $(Q_n|_{\mathrm{H}_0^1(\Omega)})_n$ converges pointwise to the identity $I : \mathrm{H}_0^1(\Omega) \to \mathrm{H}^{-1}(\Omega)$. This, by density of $\mathrm{H}_0^1(\Omega)$ into Z' and uniform boundedness, implies that $(Q_n)_n$ is a pointwise convergent sequence in all of Z'.

On the other hand, inverting the defining relation of Q_n (see (1.10)), one infers that

$$\Delta_n^{-1} = Q_n\, \mathfrak{i}' \Delta^{-1}\,.$$

We are in a position to conclude; for one, notice that $\mathfrak{i}'\Delta^{-1} : \mathrm{H}^{-1}(\Omega) \to \mathrm{Z}'$ is a compact operator. For another, $(Q_n)_n$ is a uniformly bounded, pointwise convergence sequence, and hence their composition Δ_n^{-1} converges in operator norm. □

Remark 1.2. The need of a compact inclusion from $\mathrm{H}_0^1(\Omega)$ into a larger spaces is evidenced in the previous proof. Indeed, if one tries to use the compact inclusion into $\mathrm{L}^2(\Omega)$ or $\mathrm{H}^{-1}(\Omega)$, the definition of Q_n is not natural nor well-defined anymore, as we would not have the extension Δ'.

Remark 1.3. In [Ven26], the author proved and used Proposition 1.1 to study the eigenmode convergence of the discretization of **any** symmetric Fredholm operator. Indeed, the ideas here presented were extracted from that work (see [Ven26, Chapter 2, pp. 39-43] and [Ven26, Chapter 4]), and the given example constitutes nothing but a particular case of the theory there developed.

References

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[Bre11] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011, pp. xiv+599. ISBN: 978-0-387-70913-0.

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